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On the Applicability of the Weibull Distribution to Model Annealed Glass Strength and Future Research Needs

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The applicability of the Weibull distribution to model the strength of glass, the existence of a size effect on the strength and the need for a non-destructive testing of the strength are discussed and reviewed. There are a growing number of studies that put into question the applicability of the Weibull distribution to model annealed glass fracture data. A recent study indicates that the breakage stresses are uncorrelated with the surface area, in violation of the size effect which entails the Weibull model. It is shown in this paper, however, that there is a size effect, as evidenced in an objective way by hypothesis testing using the likelihood ratio statistic. In numeric simulations it is shown that, given sample sizes of 30 specimens and the Weibull distribution being assumed, it is necessary to employ specimens which vary in surface area by a factor of about two at least, in order to detect the size effect with a success rate in excess of 0.95. To increase the use of soda-lime-silica glass in load-bearing components, however, there is a need for non-destructive testing methods, such as non-linear ultrasonic techniques. Non-destructive testing could be used not only to single out the weakest glass panes during manufacture, thus decreasing the variation in strength among as-received specimens, but also by other parties in the construction sector, such as in routine inspections on-site. Suitable stochastic models can be used together with such testing methods to develop a non-destructive strength grading of glass products.

Keywords: Glass, Weibull probability distribution, size effect, likelihood ratio test, ultrasonic materials testing

1. Introduction

The coherent and safe design of load-bearing structures made from glass is challenged by the enormous variability in the experimentally determined values of the breakage stress, cf. Table 1. Glass strength is moreover time-dependent, prone to deteriorate in wet environments and stressed conditions. In the absence of a dependable and highly repeatable value for the failure stress, it becomes necessary to employ a stochastic model for the strength. Historically, glass fracture stresses were predicted assuming a Normal probability distribution with a coefficient of variation (COV) of about 0.20, as in the early Pilkington (1939) design charts (Calderone 2000). This original use of the normal probability distribution probably owes to the circumstance that in the 19th and early 20th centuries, the variability and random errors associated with an experiment were generally considered to be of the Normal type. By the 1980s, the Weibull probability distribution had found application in the American standard ASTM E 1300-89 through its previous use in the Failure Prediction Model (FPM) of Beason and Morgan (1984). In Europe, the Weibull probability distribution was being employed in models of glass failure stresses too, cf. Sedlacek et al. (1995), Haldimann (2006). At present, there is a draft for a European standard, prEN 16612:2013, which assumes a Weibull distribution for the fitting of fracture data from experimental test results.

There are a growing number of studies which put into question the applicability of the Weibull distribution to model the strength of structural glass. Some of these studies are briefly reviewed in the following. Moreover, there is a recent study which, at first sight, appears to contradict the size effect which entails the application of the Weibull distribution. The results of that study are scrutinized and reviewed. Furthermore, the following question is investigated: Supposing that the Weibull probability distribution applies to structural glass fracture data, how likely is it thus to detect the size effect in a statistical hypothesis test? The answer to this depends, of course, on the sample sizes, the difference in population surface areas and the type of statistical test in use. Finally, the future of glass structural design is discussed in relation to ongoing research being carried out at the Division of Structural Mechanics, Lund University.

2. The Weibull Distribution

Weibull (1939) proposed a distribution function for the strength of a brittle, homogenous body in uniform tension based on the weakest link theory (WLT). About a decade earlier, Peirce (1926) had formulated the WLT in connection with the tensile strength of cotton yarn. The WLT states that the strength of a tensioned chain is governed by its weakest link, provided that the strengths of the individual links are independent of each other. The assumption of the WLT implies a size effect such that a large body is more prone to fail than a small one. Let X be the random variable for the strength of a brittle body. The cumulative distribution function of X is given by (Weibull 1939):

Table 1: Range of values for the apparent strength of glass

Source	Annealed glass strength [MPa]	COV [%]	Comment
Veer et al. (2009)	21-59	25	4PB
Veer and Rodichev (2011)	32-100	30-50	4PB
Veer (2007)	29-65	10-40	3PB/4PB
Huerta et al. (2011)	36-149	13-36	CDR
Ibid.	47-71		4PB
Overend et al. (2007)	90-165		CDR
Ritter et al. (1985)	45-200		4PB
Calderone (2000)	16-121	15-42	Uniform lateral pressure
Haldimann (2006)	34-194		CDR
Fink (2000) according to Haldimann (2006)	40-135		CDR
Carre (2010)	30-77		CDR
Nurhuda et al. (2010)	40-85		Uniform lateral pressure
This study	26-84	20	4PB

$$F_X(x) = 1 - \exp\left[-\left(\frac{x}{x_0}\right)^m\right], \quad x \geq 0 \tag{1}$$

where x_0 and m are the scale and shape parameters, respectively. The scale parameter x_0 is also seen to be the 63rd percentile in the distribution of X (Wachtman et al., 2009). The parameters may be estimated with the Maximum likelihood method, cf. Munz and Fett (1999). At equal probabilities of failure, the effect on the strength at increasing the body size from S_1 to S_2 is given by

$$\frac{x_1}{x_2} = \left(\frac{S_2}{S_1}\right)^{\frac{1}{m}} \tag{2}$$

where x_1 and x_2 are the associated values of the strengths. For instance, the implied strength of a specimen whose size is doubled is only around 0.9 of that of the original size, assuming a shape parameter $m = 6$. Weibull (1939) also proposed the three-parameter distribution

$$F_X(x) = 1 - \exp\left[-\left(\frac{x - x_u}{x_0}\right)^m\right], \quad x \geq 0 \tag{3}$$

where the third parameter x_u is a threshold value for the strength. For an identical data sample, the three-parameter distribution produces different estimates for the scale and shape parameters than the two-parameter distribution would; in general, it holds that $m' \leq m$ if m' is the three-parameter modulus and m is the two-parameter modulus (Danzer 1992). As the Weibull modulus increases in value, the variance of X decreases. Simulations with the Monte Carlo method have shown that the estimate of the third parameter, x_u, depends to a large extent on the sample being used (Danzer et al. 2007). In common measurements of glass strength, the sample sizes are limited and x_u is highly unstable. Interestingly, Warren (2001) makes the point, based on his own calculations and earlier research by Matthewson (1985), that for the three-parameter Weibull distribution, the more sensible ordinate to use in the data plot would be $\log(x^m - x_u^m)$ rather than the usual $\log(x - x_u)$. This seems to be a neglected observation in the application of the Weibull model on glass strength. Lu et al. (2002) recommend the use of the two-parameter distribution instead of the three-parameter one.

For a non-uniform stress field, Weibull (1939) suggested to integrate over the body

$$F_X(x) = 1 - \exp\left[-\frac{1}{S_0} \int_S \left(\frac{x(S)}{x_0} \right)^m \mathrm{d}S \right], \quad x \geq 0 \tag{4}$$

where S_0 denotes the size of the body. However, as Afferrante et al. (2006) point out, there is something speculative about this form of the weakest link theory, since it assumes that infinitesimal volumes of material containing cracks are non-interacting with each other.

The effect of unequal principal stresses may be accounted for by the incorporation of a location-dependent biaxial stress correction factor, as in the FPM by Beason and Morgan (1984). According to the numerical simulations of Nurhuda et al. (2009) and earlier research by Oh et al. (2003), a higher applied stress is required to break glass in the biaxial condition as compared to the uniaxial condition. Nurhuda et al. (2009) suggest a magnification factor by which the biaxial stresses can be converted into equivalent uniaxial stresses at failure.

Freudenthal (1968) showed that as long as the flaws in a brittle, homogenous body do not interact, the distribution of strength is determined by the mean number of critical flaws. By modelling a flaw as a crack with a well-defined geometry, cf. Fig. 1, and by application of a failure criterion, e.g. the one of Griffith (1920), a critical flaw may be related to the local stress field. A flaw is thus critical insofar as the corresponding crack depth leads to local fracture at the nominal stress x. The distribution function is given by

$$F_X(x) = 1 - \exp\left[-N_\mathrm{c}(x) \right], \quad x \geq 0 \tag{5}$$

where $N_\mathrm{c}(x)$ denotes the mean number of critical flaws for a particular body.

Jayatilaka and Trustrum (1977) demonstrated that the Weibull modulus m can be granted a physical meaning as a true material constant, rather than just an empirical factor. This is due to the fact that the Weibull distribution itself is generated from a flaw distribution with a monotonously decreasing frequency of crack depths, the distribution of which may be represented by a power-law by application of the mathematical theory of extreme values (Haldimann 2006).

3. Subcritical crack growth

The measured strength of glass is highly dependent on the duration of the applied load, as was noted early on by Grenet (1899). Baker and Preston (1946a) carried out experiments in which they found that the observed failure stress appeared to be about three times as great for load durations which were a fraction of a second compared to load durations of 24 hours. Other studies demonstrated that the tensile strength is increased by the removal of moisture (Baker and Preston 1946b; Culf 1957). Charles (1958a) undertook extensive tests of small rods of soda-lime-silica glass measuring the delayed time to failure under a range of (constant) stresses and temperatures in an atmosphere of saturated water vapour. It was found that the weakening of glass under prolonged stress can be attributed to the subcritical growth of surface flaws in the presence of water moisture in combination with a dependence on the temperature and pH-value of the environment. Charles (1958b) tied it all together neatly in an equation linking the subcritical growth velocity, v, to a power of the crack tip stress, σ_n, along with an exponential dependence on the temperature, akin to a thermally activated chemical process in the likes of Arrhenius (1889). The

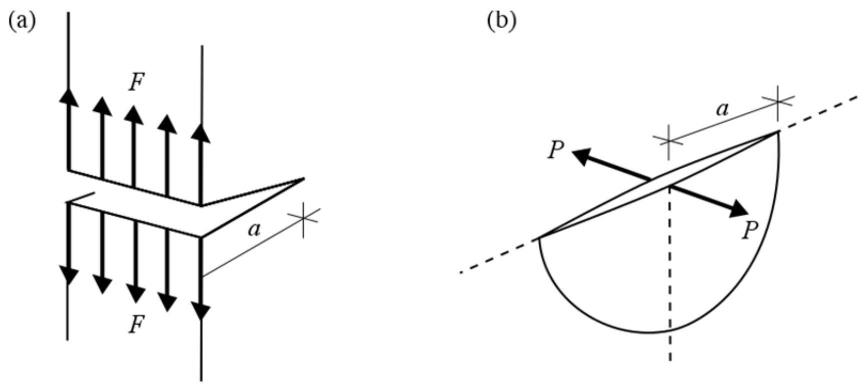

Fig. 1 A crack is an idealisation of a defect that assumes a well-defined geometry.
The figure illustrates a straight edge crack and a surface half-penny crack. Image adopted from Lawn (1993).

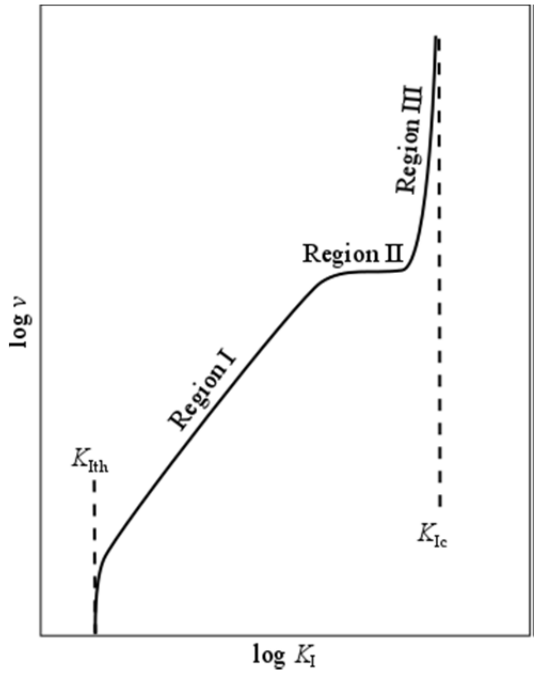

Fig. 2 Principal relationship between the crack growth velocity v and the Mode I stress intensity factor K_{I}.
Image adopted from Munz and Fett (1999).

idea was further developed in Charles and Hillig (1962).

The experiments of Wiederhorn (1967) and Wiederhorn and Bolz (1970) confirmed the crack growth velocity's exponential dependence on the crack tip stress and temperature, as well as the nearly linear dependence on the relative humidity. Figure 2 illustrates the crack growth velocity's dependence on the stress in terms of the Mode I crack intensity factor, K_{I}, which is proportional to the crack tip stress and, moreover, independent of the specimen geometry (Irwin 1957). The character of the subcritical crack growth may be divided into three regions depending on the magnitude of the stress intensity. Typically, in soda-lime glass, there exists a threshold value for the subcritical crack growth, i.e. below a certain stress intensity, about 0.25 $\mathrm{MPa}\sqrt{\mathrm{m}}$, no crack growth occurs, or at least cannot be detected (Haldimann 2006). As the stress intensity approaches a critical value, K_{Ic}, about 0.75 $\mathrm{MPa}\sqrt{\mathrm{m}}$, the crack growth becomes highly unstable and dynamic in nature, assuming velocities in the range of $\mathrm{m/s}$ to $\mathrm{km/s}$ (Lawn 1993). As a simplification for Region I (cf. Figure 2), and for fixed values of the relative humidity, temperature and pH, we write (cf. Danzer (1994))

$$v = v_0 \left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}} \right)^n \tag{6}$$

where v_0 and n are constants, n typically assuming a value of 16. Through substitution of v for $\mathrm{d}a/\mathrm{d}t$ in Eq. (6), assuming n and v_0 to be independent of time, one may derive an expression for the cumulative effect of the crack opening stress over time, as did Brown (1972). By employing the fracture criterion of Griffith (1920) and under the simplifying assumption that the crack depth at failure is significantly larger than the initial depth, a_i, it is found

$$\int_0^{t_f} x^n(t)\,\mathrm{d}t = \frac{2}{(n-2)\cdot v_0 \cdot K_{\mathrm{Ic}}^{-n} \cdot (Y\sqrt{\pi})^n \cdot a_i^{\frac{n-2}{2}}} \tag{7}$$

where t_f is the time until failure and Y is the geometry factor of the surface crack. It has been noted, that whereas the exponent n acts on the nominal stress in Brown's integral, Eq. (7), it acts on the crack tip stress in the model of Charles (1958b) (Reid 1991). With the use of Brown's integral, Eq. (7), two stress histories, $x_1(\tau_1)$ and $x_2(\tau_2)$, may be related to each other inasmuch as they cause the same cumulative effect

$$\int_0^{t_1} x_1^n(\tau_1)\,\mathrm{d}\tau_1 = \int_0^{t_2} x_2^n(\tau_2)\,\mathrm{d}\tau_2 \tag{8}$$

In particular, any stress history may be related to a constant stress acting under a certain time t, e.g. 60 seconds. In this sense, failure stresses obtained in ramp load testing may be converted into time equivalent constant stresses. One has to bear in mind, however, given the underlying assumptions, that the use of Brown's integral, Eq. (7), does not readily apply itself to breakage stress values obtained in inert or near-inert conditions, seeing as in such cases there is no, or very limited, subcritical crack growth. Moreover, when the load duration until failure is very short, as the findings of Haldimann (2006) suggest, there might be very limited subcritical crack growth, too, such that the assumptions underlying Brown's integral, Eq. (7), are violated. Reid (1991) notes that the general application of Brown's integral, Eq. (7), to arbitrary stress histories, as it occurs in the FPM of Beason and Morgan (1984), for instance, lends itself to certain contradictions, because it implies that the probability of failure at or before the time t is effectively independent of the applied stress at that same time. Haldimann (2006) elaborates on the implications of Eq. (6) in a lifetime prediction model for glass. Finally, it is well worth noting that mechanical fatigue due to cyclic loading is nonexistent in soda-lime glass, as was verified experimentally by Lü (1997).

4. Empiric Rebuttals

To begin with, a number of studies indicate that the Weibull distribution makes no better a fit to the fracture data than a Lognormal or even a Normal distribution (Lü 1997; Calderone et al. 2001; Veer et al. 2009; Huerta et al. 2011). This is confirmed by experiments carried out in the year 2013 at the Division of Structural Mechanics, Lund University, in which two series of in total 75 glass plates with the size $1000 \times 100 \times 8\ \mathrm{mm}^3$ were tested in four-point bending (cf. Section 7). The 60-second time-equivalent breakage stresses are illustrated in a Weibull probability plot beside a Lognormal probability plot in Figure 3. The associated model parameter estimates are given in Table 2. From an ocular inspection of the two probability plots, it is not evident that the Weibull distribution makes any a better fit than the Lognormal distribution. The Kolmogorov-Smirnov test provides a more objective measure of the goodness-of-fit. In a test at the significance level $\alpha = 0.05$ of the null hypothesis that the sample data are observations from populations with a Weibull distribution on one hand, and a Lognormal distribution on the other, however, no rejection is achieved; the calculated p-values are given in Table 3. Therefore, either of the fitted models might be valid. In particular, it cannot be ruled out, that the Lognormal distribution provides a superior model compared to the Weibull distribution. Furthermore, experiments suggest that the Weibull

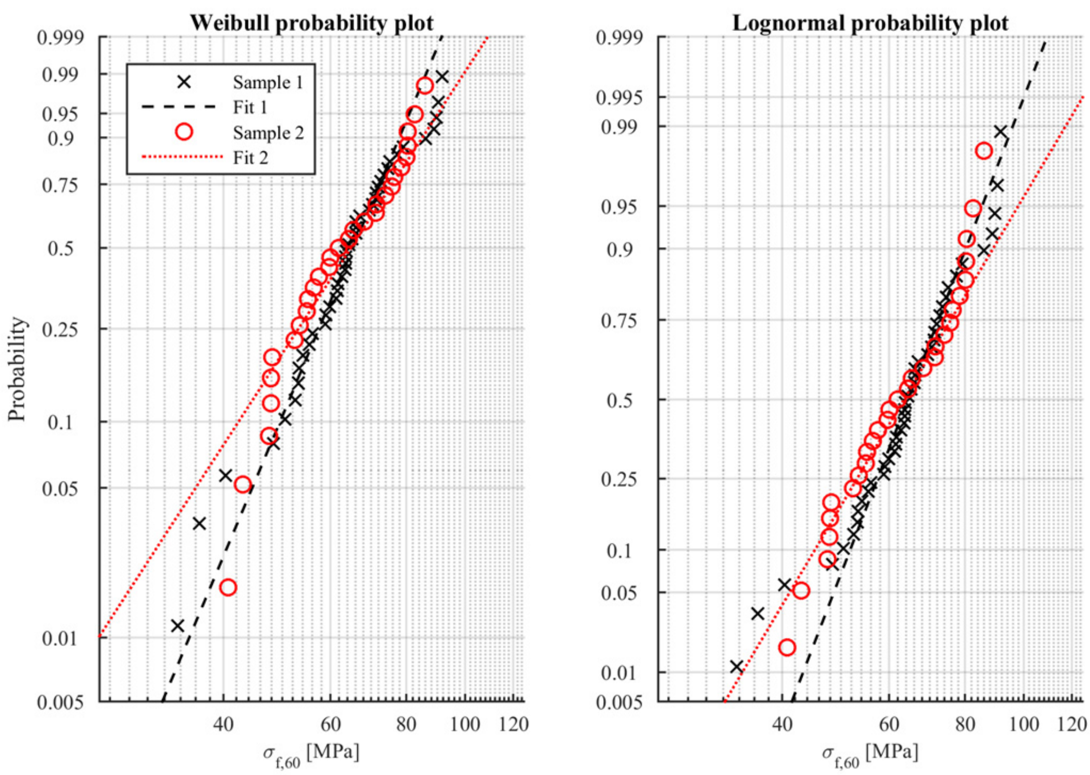

Fig. 3 Time-equivalent fracture data (60 second) from experiments conducted on glass plates in four-point bending, carried out in the year 2013 at the Division of Structural Mechanics, Lund University.

modulus m, contrary to being a material constant in the sense of Jayatilaka and Trustrum (1977), is in fact a function of a number of factors. The type of bending, i.e. the type of loading and support conditions e.g. three- and four-point or coaxial double ring bending, has a bearing on the estimate of the shape parameter (Huerta et al. 2011). The specimen geometry, i.e. the thickness, surface area and aspect ratio, have an apparent impact on the value of m (Nurhuda et al. 2010). The modulus appears to vary depending on the manufacturing processes, in particular the edge cutting, grinding and polishing methods employed (Carre 2010; Veer and Rodichev 2011; Veer 2007). This is a noteworthy point, considering the crucial impact that the edges have on glass failure stresses. Ritter et al. (1985) reports that about 80% of failures in some 2000 four-point bending tests were initiated from the edges, as evidenced from the fractographic observations of the flaw origins. In most studies, there is a more or less pronounced bimodality in the probability plots, such that the shape parameter, by all appearances, cannot be taken for a material constant over the range of stress values assumed (see e.g. Veer et al. (2009)). Some studies indicate a dependence on the state of the glass, whether it is new and as-received or weathered, having been removed from an old building and subsequently tested (Beason 1980; Calderone 2000). Furthermore, it has been claimed that the sample size itself has an impact on the estimate of m (Calderone et al. 2001). The value of the modulus seems to depend on the relative humidity of the environment (Ritter et al. 1985). The shape parameter is a function of the crack size and crack interspace (ligament size) distribution in numeric simulations of co-linear cracks in plates under remote uniform tension (Afferrante et al. 2006). There is an enormous variability in the reported estimates for m. Table 4 contains a number of these estimates, including two resulting from the experiments conducted in the year 2013 at the Division of Structural Mechanics, Lund University. Furthermore, results show that the strength of glass in the biaxial condition, when measured in terms of the maximum principal stress, is higher than in the uniaxial state (Nurhuda et al. 2009). The question is how to incorporate the effect of the multi-axial stress state into the Weibull model; some attempts have been made (Beason and Morgan 1984; Haldimann 2006). There is also the question of the potential, but mostly neglected, influence of shear stresses on the strength. Reid (2007) reports of a series of coaxial double ring bending tests with an unexpectedly high number of failures located outside the loading ring, a result which could perhaps be explained if taking into account the shear stresses. Finally, there is the issue of the theoretically implied size effect, which entails the application of the Weibull distribution due to the underlying assumption of the WLT. At least one study indicates a contradiction of the size effect in glass (Calderone et al. 2001).

Table 2: Model Parameter Estimates

	No. of spec's	Area in highest stress [$\times 10^{-3}$ m^2]	Weibull fit	Lognormal fit
Sample 1	44	30	$x_0 = 70.4$ MPa, $m = 5.46$	$\mu = 4.15$, $\sigma = 0.219$
Sample 2	29	45	$x_0 = 68.8$ MPa, $m = 5.56$	$\mu = 4.13$, $\sigma = 0.214$

Table 3: P-values in the Kolmogorov-Smirnov goodness-of-fit test

	Sample 1	Sample 2
Weibull fit	0.688	0.838
Lognormal fit	0.569	0.761

Table 4: Reported estimates of the Weibull modulus m

Test Source	Comment	m
Brown (1972)	Uniform pressure tests	7.3
Beason and Morgan (1984)	Both new and in-service plates	5 – 9
Fink (2000) according to Haldimann (2006)		7.2
Ritter et al. (1985)	Inert testing	3.22 – 10.35
Ibid.	In distilled water	4.21 – 12.64
Huerta et al. (2011)	Three-parameter distribution	1.39 – 9.98
Haldimann (2006)	Inert testing in CDR bending	8.21
Overend and Zammit (2012)		6.34
Carre (2010)	4PB	5.9 – 13.8
This study	4PB	5.46 – 5.56

5. Assessing the Size Effect

Suppose that we have two random samples $x^* = (x_1, \dots, x_{n_1})$ and $y^* = (y_1, \dots, y_{n_2})$ which are observations of the independent random variables $X = (X_1, \dots, X_{n_1})$ and $Y = (Y_1, \dots, Y_{n_2})$, respectively, where for $x \geq 0$ and $y \geq 0$ the distributions of X and Y are given by

$$F_X(x) = 1 - \exp\left[-\left(\frac{x}{x_0}\right)^m\right], \quad F_Y(y) = 1 - \exp\left[-\left(\frac{y}{y_0}\right)^m\right] \tag{9}$$

We would like to test the hypothesis

$$H_0 : x_0 = y_0 = z_0 \tag{10}$$

against the alternative

$$H_1 : x_0 \neq y_0 \tag{11}$$

If the two populations have different surface areas, we expect the parameter values, x_0 and y_0, to differ to some extent, cf. Eq. (2). In order to statistically verify any significant difference in scale parameter values, the hypothesis (10) is put to the test employing the log-likelihood ratio statistic. Let therefore, $\theta = (x_0, y_0, m) \in \Theta$, the three-parameter space forming an open subset of $\mathbb{R}^3$, and let Θ_0 denote the subset of Θ satisfying the constraints in the null hypothesis (10). With the likelihood function $L(\theta)$ we write

$$L_0 = \max\{L(\theta) : \theta \in \Theta_0\} \tag{12}$$

and

$$L_1 = \max\{L(\theta) : \theta \in \Theta\} \tag{13}$$

to denote the maximum likelihoods. By definition, we have

$$L_0 = \max\left\{\prod_{i=1}^{n_1} f_X(x_i; z_0, m) \cdot \prod_{j=1}^{n_2} f_Y(y_j; z_0, m) : (z_0, m) \in \mathrm{R}^2\right\} \tag{14}$$

and

$$L_1 = \max\left\{\prod_{i=1}^{n_1} f_X(x_i; x_0, m) \cdot \prod_{j=1}^{n_2} f_Y(y_j; y_0, m) : (x_0, y_0, m) \in \mathrm{R}^3\right\} \tag{15}$$

where f_X and f_Y are the probability density functions associated with the random variables X and Y. By substituting for the expressions of f_X and f_Y in the likelihood functions, rearranging and taking the logarithms, it is found that

$$\begin{aligned} \log L(\theta_0) = m \cdot (n_1 + n_2) \log z_0 - (n_1 + n_2) \log m - (m-1) \cdot \dots \\ \dots \cdot \left(\sum_i \log x_i + \sum_j \log y_j\right) + \left(\frac{1}{z_0}\right)^m \left(\sum_i \log x_i^m + \sum_j \log y_j^m\right) \end{aligned} \tag{16}$$

and

$$\log L(\theta) = m \cdot (n_1 \cdot \log x_0 + n_2 \cdot \log y_0) - (n_1 + n_2)\log m - (m-1) \cdot \ldots$$
$$\ldots \cdot \left(\sum_i \log x_i + \sum_j \log y_j \right) + \left(\frac{1}{x_0} \right)^m \sum_i x_i^m + \left(\frac{1}{y_0} \right)^m \sum_j y_j^m \quad (17)$$

and from this the maximum log-likelihoods $\log L_0$ and $\log L_1$ are calculated. The likelihood ratio statistic is defined

$$T = 2\log\left(\frac{L_1}{L_0} \right) \quad (18)$$

Now, according to Wilk's theorem, the likelihood ratio statistic tends in distribution to a chi-squared distribution with one degree of freedom when the null hypothesis (10) is true (Young and Smith 2005)

$$T \xrightarrow{\text{d}} \chi_1^2 \quad (19)$$

Limits may be calculated within which we expect to observe the statistic T with a given degree of confidence $1 - \alpha$.

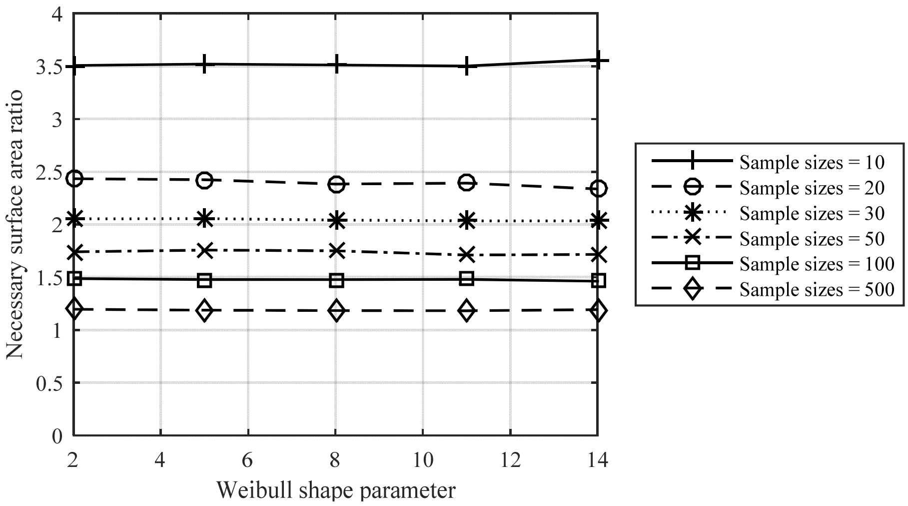

Fig. 4 The diagram illustrates the size ratio necessary in order to detect a size effect, with a high success rate, as a function of the shape parameter and the sample sizes.

6. Simulations

Given an experiment on two series of glass specimens, the specimens in the one series having a smaller surface area than the specimens in the other series, investigations have been carried out in order to determine how great a difference in the surface areas that might be required in order to detect a size effect with a high success rate, provided that such a size effect indeed exists (i.e. assuming that the Weibull distribution models the strength). With the aid of the computer software MATLAB, random samples, each of a given size, have been drawn from two Weibull populations sharing the same value of the shape parameter, m, but varying in the values of the scale parameters. Each time the samples were drawn, the hypothesis (10) was tested employing the log-likelihood ratio statistic at the significance level $\alpha = 0.05$ and the outcome (a rejection or a non-rejection) noted. By repeating this procedure 1000 times, a relative frequency for the event of a rejection of the null hypothesis (10) was determined and related to the present difference in the values of the scale parameters. As this is done for a whole range of pairs of scale parameter values, it is possible to determine how great a difference in scale parameter values that might be required in order to render a detection of the size effect with a success rate of at least 0.95 in a statistical test of size α. By means of Eq. (2) the corresponding size difference, finally, may be deducted. As can be seen from Figure 4, it would be necessary to employ specimens which vary in size by a factor of about two, at least, in order for the statistical test to detect the Weibull size effect with a rate of success greater than or equal to 0.95. Furthermore, the simulations show that the preceding statement is valid for a whole range of values of the parameter m; in particular those values which make for common estimates of m in glass experiments, viz. $2 \leq m \leq 13$ (cf. Table 4).

7. Reviewing Two Experiments

There is a recent study which indicates an absence of a size effect in structural glass. A paper by Calderone et al. (2001) reported of the results from experiments carried out on full-size rectangular window panels that ranged in size from $400 \times 2000\ \mathrm{mm}^2$ to $3000 \times 2000\ \mathrm{mm}^2$, cf. Table 5 for a specification of all the dimensions. The specimens, freely supported on four sides and stressed in uniform lateral pressure, were either of new, annealed glass or old, weathered glass, the latter having been removed from a building over 30 years old. In the study, it was indicated that there is little or no relationship between the total surface area (or indeed the surface area under the highest stress) and the breakage stress, in spite of the surface area varying by a factor of up to 7.5. The indication, if correct, would imply the violation of an assumption underlying the Weibull model. The paper mentioned doesn't contain a table for the raw data, only a diagram of all the test results plotted in unison. Nevertheless, the data in the paper have evidently been collected from the PhD Thesis of Calderone (2000). From the tables in that thesis, it is possible to reconstruct the diagram which was presented as evidence in the mentioned paper, cf. Figure 5. From an ocular inspection of the data, it may indeed seem like there is no correlation between the total surface area and the breakage stress. However, if it is assumed, *a priori*, that the Weibull distribution models the fracture stress, then a more rigorous assessment might be made of the alleged absence of a size effect. Any two samples can be compared by means of the hypothesis (10) in a likelihood ratio test. For instance, comparing the samples which correspond to the populations with the largest and smallest total surface areas, the likelihood ratio statistic $T = 8.85$ which is significant in a test of size $\alpha = 0.05$ (in fact, it is significant at $\alpha = 0.005$). As it turns out, the strength of the largest ($3000 \times 2000\ \mathrm{mm}^2$) panels is modelled by a Weibull distribution with a significantly lower value of the scale parameter as compared to most of the other panels which have surface areas only a third as large, or less in size. A statistically significant size effect is detected in many other cases too, for instance between the specimens with the third largest surface area ($1600 \times 2000\ \mathrm{mm}^2$) and the next to the smallest surface area ($500 \times 2000\ \mathrm{mm}^2$), etc. According to the simulations which have been conducted in connection with this paper, supposing that the Weibull distribution indeed models the fracture stress, and providing that the experimental data have not been confounded for some reason, the likelihood ratio test should detect the size effect with a success rate of at least 0.95 when the surface area varies by a factor of about two at least, and for sample sizes of about 30 or so specimens (cf. Section 6). Consequently, even though the failure stresses at first sight seem to be uncorrelated to the surface areas in the experiments of Calderone (2000), the data cannot be taken as counterevidence in the case of the applicability of the Weibull distribution to model glass strength.

Table 5: Specimen dimensions in the experiments of Calderone et al. (2001)

Specimen dimensions	Surface area [m^2]	Comment
$400 \times 2000\ \mathrm{mm}^2$	0.80	New, as-received
$500 \times 2000\ \mathrm{mm}^2$	1.00	– ‘’ –
$670 \times 2000\ \mathrm{mm}^2$	1.34	– ‘’ –
$2045 \times 958\ \mathrm{mm}^2$	1.96	Old, weathered
$1000 \times 2000\ \mathrm{mm}^2$	2.00	New, as-received
$1335 \times 2000\ \mathrm{mm}^2$	2.67	– ‘’ –
$1600 \times 2000\ \mathrm{mm}^2$	3.20	– ‘’ –
$2000 \times 2000\ \mathrm{mm}^2$	4.00	– ‘’ –
$3000 \times 2000\ \mathrm{mm}^2$	6.00	– ‘’ –

In the year 2013, at the Division of Structural Mechanics, Lund University, four-point bending tests were carried out on two series of glass plates with the size $1000 \times 100 \times 8\ \mathrm{mm}^3$. Series I had 45 specimens loaded by a pair of rollers spaced $300\ \mathrm{mm}$ apart, whereas the second series had 30 specimens loaded with rollers spaced $450\ \mathrm{mm}$ apart, i.e. the surface area under the highest stress varied by a factor of 1.5. The 60-second time-equivalent breakage stresses are illustrated in a Weibull probability plot in Figure 3 (cf. Section 3 concerning the time-equivalence of breakage stresses). The model parameter estimates are given in Table 2. The question is whether the data indicates a size effect at all? In a test of the null hypothesis (10), the likelihood ratio statistic $T = 0.286$, which is not significant at $\alpha = 0.05$. On the other hand, as our simulations show, a likelihood ratio test which has size $\alpha = 0.05$ is unable to detect the size effect with a power in excess of 0.95 when the surface areas vary by a factor 1.5 only, and the sample sizes are 30, assuming that the Weibull distribution indeed models the fracture data. Therefore, it is hardly possible to draw any robust conclusions about the size effect whatsoever from that experiment.

8. Discussion

As the evidence shows (cf. Table 1), a determinate value for the strength of glass is clearly outruled by the enormous scatter in the fracture data. Consequently, there is a definite need to incorporate stochastics into the modelling of the strength. Moreover, the failure stress is highly time-dependent, cf. Section 3. The application of the Weibull distribution is, however, challenged for a number of reasons. The distribution parameters appear to be a function of the type of loading, support and environmental conditions as well as the specimen geometry and manufacturing processes. The edge condition, in particular, seems to have a strong bearing on the fitted models. On one hand, edge processing tends to cause the largest flaws (Porter 2001). As the findings of Veer and Rodichev (2011) suggest, these deep flaws may prevail even after industrial edge grinding and polishing, constituting a sort of "hidden damage," as it were, seeing as the flaws tend to close up and become invisible to the eye. On the other hand, there are apparent differences among the various edge processing methods, as indicated by a strength dependence in the studies of Veer (2007) and Carre (2010). Nevertheless, the experiments show that the weakest specimens in any series of tests might just as well derive from coaxial double ring bending (with no tensile edge stressing) as from three- or four-point bending. This is telling to the fact that, at the end of the day, severe flaws may arise not just in manufacture and processing, but also in subsequent handling, transportation, maintenance, and in some cases due to vandalism. However, coaxial double ring bending produces a biaxial stress state whereas three- and four-point bending produces a uniaxial stress state. Therefore, it is not apparent, *a priori*, whether the test setup that stresses the edges is necessarily more critical or severe as compared to the setup with the biaxial stress state. It is an important question of how to incorporate the multi-axial stress state into the Weibull model (cf. EN 1288-2, EN 1288-3, EN 1288-5).

Furthermore, there is the question of whether the size effect, in practice, exists for the sort of glass components that are common in structural applications? A recent study which indicated a contradiction of the size effect in glass based on test results in Calderone (2000) has been reviewed. The data is, in fact, in support of the size effect, as evidenced by hypothesis testing using the log-likelihood ratio statistic. It therefore seems doubtful, whether any lack of a size effect may be used as counterevidence in the case of the applicability of the Weibull distribution to model the strength of glass. However, as the simulations which were carried out in connection with this paper show, it is necessary to employ specimens which vary in size by a factor of two at least, in order to detect the size effect with a high success rate, given the usual sample sizes of about 30 or so specimens, supposing that the Weibull distribution indeed models glass strength. Therefore, some experiments will hardly render any useful information at all about the alleged size effect in glass. Of course, this line of reasoning supposes that the specimens tested are nominally equal in all other aspects. This is a simplifying assumption which involves potential errors. For instance, what if the specimens have been cut out and prepared from different original jumbo plates with varying flaw populations? There might be circumstances that confound the test results insofar that we do not detect the size effect in spite of there being one. Interestingly, in a literary review of the strength of glass some 50 years ago, Sugarman (1967) stated that "[w]hereas size effects were until recently thought to be of prime importance, it is doubtful whether size plays a very significant part in the realizable strength." It is worth noting, that the stressed surface area in most EN codes is much smaller than the surface area that might be subjected to a design load in a real, practical situation. The question of whether strength in glass scales with size is of great importance, when one takes into consideration the necessary extrapolations having to be made from experiments conducted in agreement with some EN code.

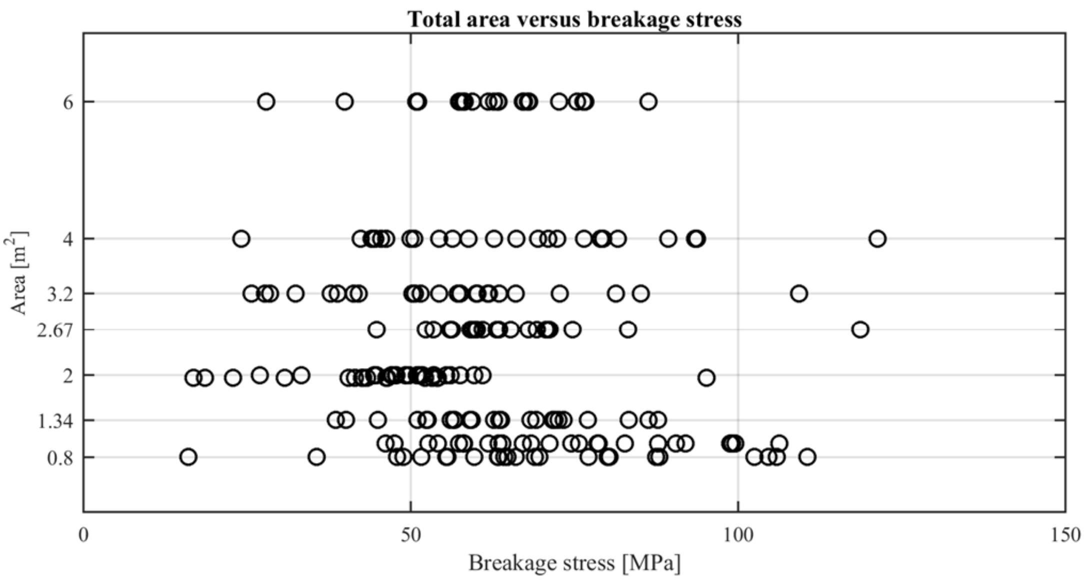

Fig. 5 Data results on glass breakage stresses versus total surface area from Calderone (2000).

More generally, there seems to be a persistence about the application of the Weibull distribution to glass fracture data, which perhaps has to do with the appeal of the physical arguments underlying the model. Haldimann (2006), for instance, readily acknowledges the enormous scatter in the published parameter estimates and gives testimony to the fact that the Lognormal or even the Normal distributions appear to fit data just as well or even better. Nonetheless, he discredits the use of the Lognormal and Normal distributions on the grounds that it would be illogical to base a model for the strength in a brittle material like glass on a distribution function that does not derive from the WLT. He makes the claim that the mismatch with data is only apparent; caused, as it were, by the confounding effect of the subcritical crack growth during testing, owing to the presence of water moisture in the environment. His experiments on five series of coaxial double ring bending tests suggests, nonetheless, that testing at high stress rates (ostensibly 20 MPa/s or more) eliminates the influence of the subcritical crack growth, i.e. mimicking the inert environment. (The idea that high rates of loading produces near-inert strength values was in principal formulated by Baker and Preston (1946a).) Unfortunately, there is nothing in Haldimann's data that lends itself to the conclusion that the scatter in the estimates of the Weibull modulus vanishes at high stress rates. The variability among the parameter estimates in the inert test results of Ritter et al. (1985) suggest, in fact, that the scatter does not vanish, cf. Table 4.

Huerta et al. (2011) also acknowledge the awkward circumstance that the estimates for m exhibit such a great variance, as seen, not least, in their own experimental data, cf. Table 4, and that a Weibull distribution appears to provide no better a fit than a Lognormal or even Normal distribution. Notwithstanding, they promote the choice of the Weibull distribution on the grounds that it harmonizes with the alleged size effect.

It is important to bear in mind, that the Weibull distribution is forgiving for fitting fracture data, as evidenced in Monte Carlo simulations of random samples from Weibull populations, conducted by Danzer et al. (2001). They found that with the usually limited sample sizes common in experiments on glass, it is possible to estimate the parameters of the Weibull distribution at a reasonable level of confidence, but it is not possible to decide if the distribution is of a Weibull-type or any other type. In fact, a Weibull distribution can be fitted to any small sample of fracture data from a brittle material (Danzer et al. 2007). Of course, this does not necessarily correspond with the population itself being of the Weibull-type. It is therefore noteworthy that when Weibull fits are compared to Lognormal or even Normal distribution fits, it is not unusual for all of them to exhibit a similarly high goodness-of-fit, as measured by the coefficients of determination in the studies of Lü (1997); Calderone et al. (2001); Veer et al. (2009); Huerta et al. (2011). When fracture data are represented in a probability plot, the strength values are ranked in ascending order. The failure probability of the ith specimen, F_i, is usually estimated as

$$F_i = \frac{i - 1/2}{N} \qquad i = 1,2,...,N \tag{20}$$

where N denotes the sample size (Munz and Fett 1999). Depending on how much the observed strength values happen to cluster together in some intervals and spread out in others, as compared to a "perfect" Weibull distribution, curved structures appear in the probability plot (Danzer et al. 2001). Moreover, due to the highly

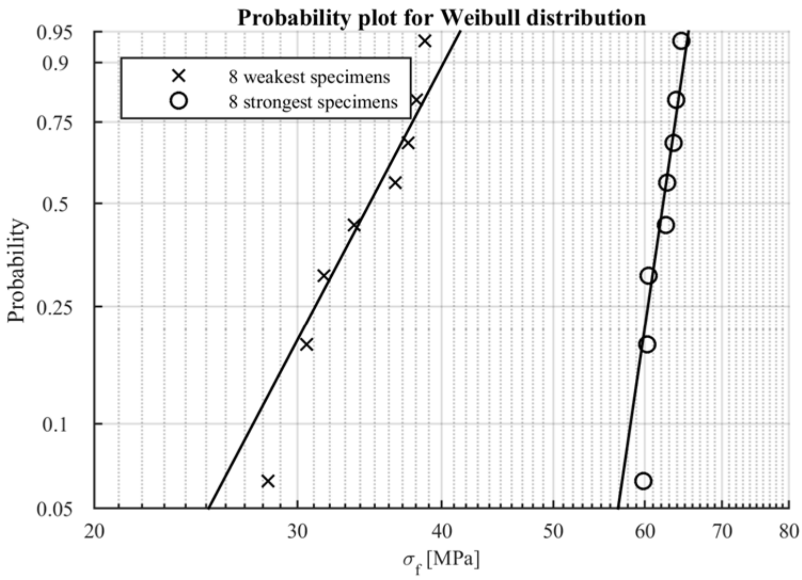

Fig. 6 In Veer (2007) the 8 weakest and 8 strongest test results were separated from 48 test results, bringing about a seemingly improved fit.

nonlinear scales in the Weibull probability plot, these statistical artefacts get magnified at small and large failure probabilities (Ibid.). It is therefore quite possible for a small data sample, e.g. of size 30 which is a common sample size in structural glass experiments, to exhibit an apparent bimodality in the probability plot owing to this sort of statistical artefact, even if the underlying population is of the Weibull-type. All this is presuming that the experimental measurements have been carried out in a coherent and consistent way. Otherwise, one may suspect that errors made in connection with the experimental method might produce peculiarities, artefacts as it were, in the probability plots, too.

Danzer et al. (2007) ran numerical simulations on random samples of size 30 from Weibull populations and made a classification into different types according to the visual impression that the samples made in an ordinary probability plot. There would be samples fitting the original population quite well, samples “apparently” fitting a population with quite different moduli or scale parameter values, samples with an “apparent” bimodality, samples “apparently” fitting a three parameter distribution, etc. In these simulations, the samples “apparently” fitting a bimodal population were about as frequent as the samples making a reasonable or good fit. If these findings may be generalized, then how can the experimenter distinguish between the statistical artefact on the one hand and the true material behaviour on the other? On a similar note, consider the tendency in the literature to interpret the curved behaviour of the tails as indications, if not evidence, of a bimodal distribution. Veer (2007), for instance, separated the eight weakest and the eight strongest test results, respectively, from 48 test results on glass with the size $1000 \times 125 \times 10$ mm^3 in four-point bending, thus bringing about a seemingly improved fit, however, with different Weibull moduli, cf. Figure 6. The separated data belongs to two different causal populations, he suggests.

9. The Future of Glass Design

The evidence available stresses the difficulties associated with any attempts of fitting a Weibull distribution to the fracture data of new, as-received glass. As an alternative, it seems promising to employ direct numeric simulations. Veer (2007) did this in the case of edge fractures. His data suggested the distribution of one type of rather grave flaws, producing a failure stress of 30 MPa with a COV of 7% in every 2 meters, and another distribution of minor flaws in every millimetre with an associated failure stress of 60 MPa and a COV of 12%. Yankelevsky (2014) was able to simulate the strength of a square glass plate, supported on its ends in uniform lateral loading, based on a few simple assumptions on part of the surface condition. E.g., he assumed that there exists a maximum limit of about $200\ \mu$m to the depth values of the flaw distribution, in essence mimicking what might be expected in an as-received jumbo plate which has undergone production control processes. In Monte Carlo simulations, he found a probability distribution for the strength of the plate, similar in shape to a Normal distribution. To begin with, the two models mentioned might be incorporated into one, perhaps with an increased capacity to address the evidently rather different flaw distributions which characterize the edges on one hand and the surfaces on the other. There is certainly a great need for further research in this direction. Neither study, for instance, addressed the long-term strength. Nevertheless, as a number of researchers have testified to, it is an extremely debilitating circumstance that failure stresses of about 20-30 MPa keep cropping up so frequently, as evidenced in the published experimental data. Interestingly, it is reported that among several hundred tests, no failure stresses below about 20 MPa were encountered (Veer 2007; Huerta et al. 2011). Regarding the long-term strength, it may be assumed to be more than halved, as suggested by the recommended value of 8 MPa by the Institution of Structural Engineers (2000) and the lifetime prediction model of Haldimann (2006), cf. Fig. 7. As long as these low strength values prevail, it really is the question of to what extent it would improve the confidence in glass components, even if a model better than the Weibull distribution were available. At least, for the application of load-bearing monolithic annealed glass components, the low failure stresses frequently observed pose a severe problem.

There is an urgent need for a non-destructive verification method for the strength in annealed glass, if the structural designer is to have any reasonable confidence in the monolithic strength assuming values significantly greater than about 20 MPa. As of today, no such non-destructive method is in practical use nor available on the market. However, at Lund University, investigations are underway, in collaboration with Acoustic Agree, for the application of a nonlinear ultrasonic technique for non-destructive determination of the amount and size of defects in glass. This non-destructive method is already in use on metals and plastics. Ultrasonic testing on glass, however, is at present only used for the measurement of the thickness of large plates using the pulse-echo technique (Krautkrämer and Krautkrämer 1990). The nonlinear ultrasonic technique, if successfully applied to glass, would bring about significant benefits in the design of load-bearing glass components. Not only could the specimens which suffer from the most severe flaws be singled out, but furthermore, due to the portable nature of the ultrasonic measuring device, assessments could be made of the strength of elements already installed in an existing structure. In the first stage of the application, it is necessary to calibrate the measuring equipment, such that for a certain pre-determined and fixed stress value, e.g. 45 MPa, the device will indicate whether a given glass component would sustain the stress or not, in (hypothetical) direct tension. For this, it is necessary to conduct a range of experiments. Such experiments are presently being planned for at the Department of Structural Mechanics, Lund University.

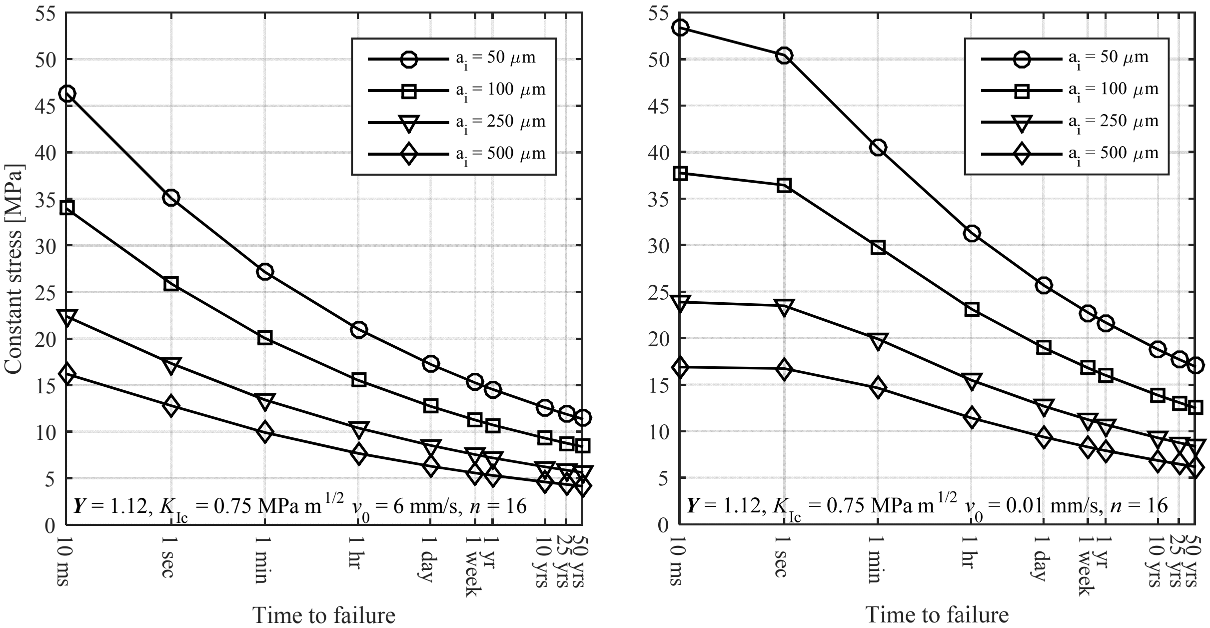

Fig. 7 The time-dependent strength of a crack for a number of different initial crack depths a_i according to the lifetime prediction model of Haldimann (2006).

10. Conclusions

The Weibull theory rests on a firm theoretical foundation including the notion of a material constant modulus m. There is a growing body of empirical evidence on the contrary to the notion of the Weibull modulus m as representing any sort of material constant. For the small sample sizes commonly employed in structural glass experiments, a Weibull distribution doesn't appear to provide any a better fit than a Lognormal distribution. A recent experiment confirms the size effect in annealed glass. According to numerical simulations on samples of size 30, it is necessary to employ specimens which vary in surface area by a factor of about two at least, if any size effect should be detected with a high rate of success by a statistical test of small size, providing that a size effect exists. In the future, ultrasonic materials testing might provide a means for singling out those glass specimens which suffer from the most severe flaws.

11. Summary

The safe design of load-bearing glass structures is challenged by the great variability in the breakage stresses. Moreover, the strength is time-dependent, prone to deteriorate in wet environments and under prolonged stress. Originally the variation in strength was predicted by use of a Normal distribution. These days, the Weibull distribution is preferred over the Normal distribution, as evidenced by its application in the American standards and the draft for a European standard. However, as a growing number of studies show, the application of the Weibull distribution on annealed glass fracture data is questionable for a number of reasons, viz. the goodness-of-fit is not superior compared to a Lognormal or even a Normal distribution, the shape parameter estimates exhibit an enormous variability depending, by all appearances, on the type of loading, the specimen geometry, manufacturing processes etc. A recent study nevertheless confirms the existance of a size effect in annealed window glass under uniform lateral pressure. It seems doubtful, whether any lack of a size effect may be used as counterevidence in the case of the applicability of the Weibull distribution. According to numerical simulations, it is necessary to employ specimens which vary in size by a factor of about two at least, in order to detect the size effect with a high rate of success, given sample sizes of 30 specimens and a log-likelihood ratio test, assuming that the Weibull distribution models glass breakage stresses. The low failure stresses which are frequently observed in experiments on monolithic annealed glass pose a severe problem to the coherent design of load-bearing glass components. There is a need for a non-destructive verification method for the strength in annealed glass, if the structural designer is to have any reasonable confidence in the strength assuming values significantly greater than about 20 MPa. In future, it may be possible to single out the weakest specimens by use of ultrasonic testing techniques. This would bring about significant benefits to the safe design of glass structures.

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References

Afferrante, L., Ciavarella, M., Valenza, E.: Is Weibull's modulus really a material constant? Example case with interacting collinear cracks. Int J Solids Struct **43**, 5147 – 5157 (2006). doi:http://dx.doi.org/10.1016/j.ijsolstr.2005.08.002

Arrhenius, S.A.: Über die Reaktionsgeschwindigkeit bei der Inversion von Rohrzucker durch Säuren. Z. Physik. Chem. **4**, 226–248 (1889)

ASTM 1300-89: Practice to Determine the Minimum Thickness of Annealed Glass Required to Resist a Specified Load. American Society for Testing Materials (1989)

Baker, T.C., Preston, F.W.: The Effect of Water on the Strength of Glass. J Appl Phys **17**, 179–188 (1946a). doi:http://dx.doi.org/10.1063/1.1707703

Baker, T.C., Preston, F.W.: Fatigue of Glass under Static Loads. J Appl Phys **17**, 170–178 (1946b). doi:http://dx.doi.org/10.1063/1.1707702

Beason, W.L.: A Failure Prediction Model for Window Glass. NTIS Accession no. PB81-148421, Texas Tech University, Institute for Disaster Research, (1980)

Beason, W.L., Morgan, J.R.: Glass Failure Prediction Model. Journal of Structural Engineering **110**, 197–212 (1984)

Brown, W.G.: A load duration theory for glass design. In: NRCC 12354. pp. 75–78. National Research Council of Canada, Ottawa, (1972)

Calderone, I.J.: The Equivalent Wind Load for Window Glass Design. PhD Thesis, Monash University, Australia (2000)

Calderone, I.J., MacDonald, C.M., Jacob, L., Jacob and Associates Pty Ltd: The Fallacy of the Weibull Distribution for Window Glass Design. In: Glass Performance Days 2001, Tampere, Finland, June 2001, pp. 293–297

Carre, H.: Etude du comportement à la rupture d'un matériau fragile précontraint : le verre trempé. PhD Thesis, Ecole Nationale des Ponts et Chaussées (2010)

Charles, R.J.: Static Fatigue of Glass I. J Appl Phys **29**, 1549–1553 (1958a)

Charles, R.J.: Static Fatigue of Glass II. J Appl Phys **29**, 1554–1560 (1958b)

Charles, R.J., Hillig, W.B.: The kinetics of glass failure by stress corrosion. In: Symposium on Mechanical Strength of Glass and Ways of Improving it, Charleroy, Belgium 1962, pp. 511–527

Culf, C.J.: Fracture of Glass under Various Liquids and Gases. J. Soc. Glass Tech. **41**, 157 (1957)

Danzer, R.: A General Strength Distribution Function for Brittle Materials. J Eur Ceram Soc **10**, 461–472 (1992)

Danzer, R.: Ceramics: Mechanical Performance and Lifetime Prediction. The Encyclopedia of Advanced Materials **1**, 385–398 (1994)

Danzer, R., Lube, T., Supancic, P.: Monte Carlo Simulations of Strength Distributions of Brittle Materials - Type of Distribution, Specimen and Sample Size. Z. Metallkd. **92**, 773–783 (2001)

Danzer, R., Supancic, P., Pascual Herrero, J., Lube, T.: Fracture Statistics of Ceramics - Weibull Statistics and Deviations from Weibull Statistics. Eng Fract Mech **74**, 2919–2932 (2007). doi:10.1016/j.engfracmech.2006.05.028

EN 1288-2:2000: Glass in building – Determination of the bending strength of glass – Part 2: Coaxial double ring test on flat specimens with large test surface areas. CEN (2000)

EN 1288-3:2000: Glass in building – Determination of the bending strength of glass – Part 3: Test with specimen supported at two points (four point bending). CEN (2000)

EN 1288-5:2000: Glass in building – Determination of the bending strength of glass – Part 5: Coaxial double ring test on flat specimens with small test surface areas. CEN (2000)

Fink, A.: Ein Beitrag zum Einsatz von Floatglas als dauerhaft tragender Konstruktionswerkstoff im Bauwesen. PhD Thesis, Technische Hochschule Darmstadt (2000)

Freudenthal, A.M.: Statistical Approach to Brittle Fracture. In: Liebowitz, H. (ed.), vol. II. Fracture. Academic Press, New York (1968)

Grenet, L.: Mechanical Strength of Glass. Bull. Soc. Enc. Industr. Nat. Paris **4**, 838–848 (1899)

Griffith, A.A.: The Phenomena of Rupture and Flow in Solids. Phil. Trans. R. Soc. **A221**, 163 (1920)

Haldimann, M.: Fracture strength of structural glass elements – analytical and numerical modelling, testing and design. PhD Thesis, EPFL (2006)

Huerta, M.C., Pacios-Alvarez, A., Lamela-Rey, M.J., Fernández-Canteli, A.: Influence of experimental test type on the determination of probabilistic stress distribution. In: Glass Performance Days 2011, Tampere, Finland 2011

Institution of Structural Engineers: Structural use of glass in buildings. London (2000)

Irwin, G.R.: Analysis of Stresses and Strains Near the End of a Crack Traversing a Plate. J. Appl. Mech. **24**, 361 (1957)

Jayatilaka, A.D.S., Trustrum, K.: Statistical approach to brittle fracture. J Mater Sci **12**, 1426–1430 (1977)

Krautkrämer, J., Krautkrämer, H.: Ultrasonic testing of materials, 4th ed. Springer-Verlag, (1990)

Lawn, B.: Fracture of Brittle Solids, 2nd ed. Cambridge Solid State Science Series. Cambridge University Press, Cambridge, UK (1993)

Lu, C., Danzer, R., Fischer, F.D.: Fracture statistics of brittle materials: Weibull or normal distribution. Phys. Rev. E **65**, 067102 (2002). doi:10.1103/PhysRevE.65.067102

Lü, B.-T.: Fatigue strength prediction of soda-lime glass. Theor Appl Fract Mec **27**, 107–114 (1997)

Matthewson, M.J.: An investigation of the statistics of fracture. In: Kurkijian, C.R. (ed.) Strength of Inorganic Glasses. pp. 429–442. Plenum Press, (1985)

Munz, D., Fett, T.: Ceramics. Mechanical Properties, Failure Behaviour, Materials Selection. Materials Science, vol. 36. Springer-Verlag Berlin Heidelberg New York, (1999)

Nurhuda, I., Lam, N.T.K., Gad, E.F.: The statistical distribution of the strength of glass. In: Aravinthan, T. (ed.) Futures in mechanics of structures and materials: proceedings of the 20th Australasian Conference on the Mechanics of Structures and Materials, Toowoomba, Australia, 2-5 December 2008. CRC Press, Boca Raton, Fla. (2009)

Nurhuda, I., Lam, N.T.K., Gad, E.F., Calderone, I.: Estimation of strengths in large annealed glass panels. Int J Solids Struct **47**, 2591 – 2599 (2010). doi:http://dx.doi.org/10.1016/j.ijsolstr.2010.05.015

Oh, S.Y., Shin, H.S., Suh, C.M.: Evaluation of Biaxial Bending Strength in Damaged Soda Lime Glass. Int J Mod Phys B **17**, 1329–1334 (2003)

Overend, M., Parke, G.A.R., Buhagiar, D.: Predicting Failure in Glass - A General Crack Growth Model. Journal of Structural Engineering **133**, 1146–1155 (2007)

Overend, M., Zammit, K.: A computer algorithm for determining the tensile strength of float glass. Eng Struct **45**, 68 – 77 (2012). doi:http://dx.doi.org/10.1016/j.engstruct.2012.05.039

Peirce, F.T. J. of Textile Inst. Trans. **17**, 355 (1926)

Pilkington Brothers Ltd. U.K.: Stress and strength calculations in flat glass. Internal Pilkington Design Guidelines (1939)

Porter, M.: Aspects of Structural Design with Glass. PhD Thesis, The University of Oxford (2001)

prEN 16612: Glass in building - Determination of the load resistance of glass panes by calculation and testing. CEN (2013)

Reid, S.G.: Flaws in the Failure Prediction Model of Glass Strength. In: Sixth International Conference on Applications of Statistics and Probability in Civil Engineering, Mexico City, Mexico, June 1991, pp. 111–117 (1991)

Reid, S.G.: Effects of spatial variability of glass strength in ring-on-ring tests. Civil Engineering and Environmental Systems **24**, 139–148 (2007)

Ritter, J.E., Service, T.H., Guillemet, C.: Strength and fatigue parameters for soda-lime glass. Glass Technol **26**, 273–278 (1985)

Sedlacek, G., Blank, K., Güsgen, J.: Glass in Structural Engineering. The Structural Engineer **73**, 17–22 (1995)

Sugarman, B.: Strength of Glass (A Review). J Mater Sci **2**, 275–283 (1967)

The MathWorks Inc.: MatLab R2015a. (2015)

Wachtman, J.B., Cannon, W.R., Matthewson, M.J.: Mechanical properties of ceramics. Wiley, Hoboken, N.J. (2009)

Warren, P.D.: Fracture of brittle materials: effects of test method and threshold stress on the Weibull modulus. J Eur Ceram Soc **21**, 335–342 (2001)

Veer, F.A.: The strength of glass, a nontransparent value. HERON **52**, 87–104 (2007)

Veer, F.A., Louter, C., Bos, F.P.: The strength of annealed, heat-strengthened and fully tempered float glass. Fatigue Fract Eng M **32**, 18–25 (2009). doi:10.1111/j.1460-2695.2008.01308.x

Veer, F.A., Rodichev, Y.M.: The structural strength of glass: Hidden damage. Strength Mater+ **43**, 302–315 (2011)

Weibull, W.: A Statistical Theory of the Strength of Materials. Ingenjörsvetenskapsakademiens handlingar **151** (1939)

Wiederhorn, S.M.: Influence of Water Vapor on Crack Propagation in Soda-Lime Glass. J. Am. Ceram. Soc. **50**, 407–417 (1967)

Wiederhorn, S.M., Bolz, L.H.: Stress corrosion and static fatigue of glass. J. Am. Ceram. Soc. **53**, 543–548 (1970)

Yankelevsky, D.Z.: Strength prediction of annealed glass plates – A new model. Eng Struct **79**, 244 – 255 (2014). doi:http://dx.doi.org/10.1016/j.engstruct.2014.08.017

Young, G.A., Smith, R.L.: Essentials of Statistical Inference. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, (2005)